

Concept definitions from *Elements of Programming*

Alexander Stepanov Paul McJones

January 16, 2017

Introduction

This is a summary of the concept definitions from *Elements of Programming*, published by Addison-Wesley Professional in June 2009. For more information, see www.elementsofprogramming.com.

Chapter 1: Foundations

Regular(T) \triangleq

T's computational basis includes equality, assignment, destructor, default constructor, copy constructor, total ordering (or default total ordering) and underlying type.

FunctionalProcedure(F) \triangleq

F is a *regular* procedure defined on regular types: replacing its inputs with equal objects results in equal output objects.

UnaryFunction(F) \triangleq

FunctionalProcedure(F)

\wedge *Arity*(F) = 1

\wedge *Domain* : *UnaryFunction* \rightarrow *Regular*

F \mapsto *InputType*(F, 0)

HomogeneousFunction(F) \triangleq

FunctionalProcedure(F)

\wedge *Arity*(F) > 0

\wedge ($\forall i, j \in \mathbb{N}$)($i, j < \text{Arity}(F) \Rightarrow (\text{InputType}(F, i) = \text{InputType}(F, j))$)

\wedge *Domain* : *HomogeneousFunction* \rightarrow *Regular*

F \mapsto *InputType*(F, 0)

property(F : *UnaryFunction*)

`regular_unary_function` : F

$$f \mapsto (\forall f' \in F)(\forall x, x' \in \text{Domain}(F)) \\ (f = f' \wedge x = x') \Rightarrow (f(x) = f'(x'))$$

Chapter 2: Transformations and Their Orbits

$$\text{Predicate}(P) \triangleq \\ \text{FunctionalProcedure}(P) \\ \wedge \text{Codomain}(P) = \text{bool}$$

$$\text{HomogeneousPredicate}(P) \triangleq \\ \text{Predicate}(P) \\ \wedge \text{HomogeneousFunction}(P)$$

$$\text{UnaryPredicate}(P) \triangleq \\ \text{Predicate}(P) \\ \wedge \text{UnaryFunction}(P)$$

$$\text{Operation}(\text{Op}) \triangleq \\ \text{HomogeneousFunction}(\text{Op}) \\ \wedge \text{Codomain}(\text{Op}) = \text{Domain}(\text{Op})$$

$$\text{Transformation}(F) \triangleq \\ \text{Operation}(F) \\ \wedge \text{UnaryFunction}(F) \\ \wedge \text{DistanceType} : \text{Transformation} \rightarrow \text{Integer}$$

Chapter 3: Associative Operations

$$\text{BinaryOperation}(\text{Op}) \triangleq \\ \text{Operation}(\text{Op}) \\ \wedge \text{Arity}(\text{Op}) = 2$$

property(Op : BinaryOperation)

associative : Op

$$\text{op} \mapsto (\forall a, b, c \in \text{Domain}(\text{op})) \text{op}(\text{op}(a, b), c) = \text{op}(a, \text{op}(b, c))$$

$$\text{Integer}(I) \triangleq \\ \text{successor} : I \rightarrow I \\ \quad n \mapsto n + 1 \\ \wedge \text{predecessor} : I \rightarrow I \\ \quad n \mapsto n - 1 \\ \wedge \text{twice} : I \rightarrow I \\ \quad n \mapsto n + n \\ \wedge \text{half_nonnegative} : I \rightarrow I \\ \quad n \mapsto \lfloor n/2 \rfloor, \text{ where } n \geq 0$$

\wedge `binary_scale_down_nonnegative` : $I \times I \rightarrow I$
 $(n, k) \mapsto \lfloor n/2^k \rfloor$, where $n, k \geq 0$
 \wedge `binary_scale_up_nonnegative` : $I \times I \rightarrow I$
 $(n, k) \mapsto 2^k n$, where $n, k \geq 0$
 \wedge `positive` : $I \rightarrow \text{bool}$
 $n \mapsto n > 0$
 \wedge `negative` : $I \rightarrow \text{bool}$
 $n \mapsto n < 0$
 \wedge `zero` : $I \rightarrow \text{bool}$
 $n \mapsto n = 0$
 \wedge `one` : $I \rightarrow \text{bool}$
 $n \mapsto n = 1$
 \wedge `even` : $I \rightarrow \text{bool}$
 $n \mapsto (n \bmod 2) = 0$
 \wedge `odd` : $I \rightarrow \text{bool}$
 $n \mapsto (n \bmod 2) \neq 0$

Chapter 4: Linear Orderings

$\text{Relation}(\mathbf{R}) \triangleq$
 $\text{HomogeneousPredicate}(\mathbf{R})$
 $\wedge \text{Arity}(\mathbf{R}) = 2$

property($\mathbf{R} : \text{Relation}$)
`transitive` : \mathbf{R}
 $r \mapsto (\forall a, b, c \in \text{Domain}(\mathbf{R})) (r(a, b) \wedge r(b, c) \Rightarrow r(a, c))$

property($\mathbf{R} : \text{Relation}$)
`strict` : \mathbf{R}
 $r \mapsto (\forall a \in \text{Domain}(\mathbf{R})) \neg r(a, a)$

property($\mathbf{R} : \text{Relation}$)
`reflexive` : \mathbf{R}
 $r \mapsto (\forall a \in \text{Domain}(\mathbf{R})) r(a, a)$

property($\mathbf{R} : \text{Relation}$)
`symmetric` : \mathbf{R}
 $r \mapsto (\forall a, b \in \text{Domain}(\mathbf{R})) (r(a, b) \Rightarrow r(b, a))$

property($\mathbf{R} : \text{Relation}$)
`asymmetric` : \mathbf{R}
 $r \mapsto (\forall a, b \in \text{Domain}(\mathbf{R})) (r(a, b) \Rightarrow \neg r(b, a))$

property($\mathbf{R} : \text{Relation}$)
`equivalence` : \mathbf{R}
 $r \mapsto \text{transitive}(r) \wedge \text{reflexive}(r) \wedge \text{symmetric}(r)$

property($F : \text{UnaryFunction}, R : \text{Relation}$)
requires($\text{Domain}(F) = \text{Domain}(R)$)
key_function : $F \times R$
 $(f, r) \mapsto (\forall a, b \in \text{Domain}(F)) (r(a, b) \Leftrightarrow f(a) = f(b))$

property($R : \text{Relation}$)
total_ordering : R
 $r \mapsto \text{transitive}(r) \wedge$
 $(\forall a, b \in \text{Domain}(R))$ exactly one of the following holds:
 $r(a, b), r(b, a),$ or $a = b$

property($R : \text{Relation}$)
total_ordering : R
 $r \mapsto \text{transitive}(r) \wedge$
 $(\forall a, b \in \text{Domain}(R))$ exactly one of the following holds:
 $r(a, b), r(b, a),$ or $a = b$

property($R : \text{Relation}, E : \text{Relation}$) **requires**($\text{Domain}(R) = \text{Domain}(E)$)
weak_ordering : R
 $r \mapsto \text{transitive}(r) \wedge (\exists e \in E) \text{equivalence}(e) \wedge$
 $(\forall a, b \in \text{Domain}(R))$ exactly one of the following holds:
 $r(a, b), r(b, a),$ or $e(a, b)$

$\text{TotallyOrdered}(T) \triangleq$
 $\text{Regular}(T)$
 $\wedge < : T \times T \rightarrow \text{bool}$
 $\wedge \text{total_ordering}(<)$

Chapter 5: Ordered Algebraic Structures

property($T : \text{Regular}, \text{Op} : \text{BinaryOperation}$)
requires($T = \text{Domain}(\text{Op})$)
identity_element : $T \times \text{Op}$
 $(e, \text{op}) \mapsto (\forall a \in T) \text{op}(a, e) = \text{op}(e, a) = a$

property($F : \text{Transformation}, T : \text{Regular}, \text{Op} : \text{BinaryOperation}$)
requires($\text{Domain}(F) = T = \text{Domain}(\text{Op})$)
inverse_operation : $F \times T \times \text{Op}$
 $(\text{inv}, e, \text{op}) \mapsto (\forall a \in T) \text{op}(a, \text{inv}(a)) = \text{op}(\text{inv}(a), a) = e$

property($\text{Op} : \text{BinaryOperation}$)
commutative : Op
 $\text{op} \mapsto (\forall a, b \in \text{Domain}(\text{Op})) \text{op}(a, b) = \text{op}(b, a)$

$\text{AdditiveSemigroup}(T) \triangleq$
 $\text{Regular}(T)$
 $\wedge + : T \times T \rightarrow T$

$$\begin{aligned}
& \wedge \text{associative}(+) \\
& \wedge \text{commutative}(+) \\
\text{MultiplicativeSemigroup}(\mathbb{T}) & \triangleq \\
& \text{Regular}(\mathbb{T}) \\
& \wedge \cdot : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T} \\
& \wedge \text{associative}(\cdot) \\
\text{AdditiveMonoid}(\mathbb{T}) & \triangleq \\
& \text{AdditiveSemigroup}(\mathbb{T}) \\
& \wedge 0 \in \mathbb{T} \\
& \wedge \text{identity_element}(0, +) \\
\text{MultiplicativeMonoid}(\mathbb{T}) & \triangleq \\
& \text{MultiplicativeSemigroup}(\mathbb{T}) \\
& \wedge 1 \in \mathbb{T} \\
& \wedge \text{identity_element}(1, \cdot) \\
\text{AdditiveGroup}(\mathbb{T}) & \triangleq \\
& \text{AdditiveMonoid}(\mathbb{T}) \\
& \wedge - : \mathbb{T} \rightarrow \mathbb{T} \\
& \wedge \text{inverse_operation}(\text{unary } -, 0, +) \\
& \wedge - : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T} \\
& \quad (\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} + (-\mathbf{b}) \\
\text{MultiplicativeGroup}(\mathbb{T}) & \triangleq \\
& \text{MultiplicativeMonoid}(\mathbb{T}) \\
& \wedge \text{multiplicative_inverse} : \mathbb{T} \rightarrow \mathbb{T} \\
& \wedge \text{inverse_operation}(\text{multiplicative_inverse}, 1, \cdot) \\
& \wedge / : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T} \\
& \quad (\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \cdot \text{multiplicative_inverse}(\mathbf{b}) \\
\text{Semiring}(\mathbb{T}) & \triangleq \\
& \text{AdditiveMonoid}(\mathbb{T}) \\
& \wedge \text{MultiplicativeMonoid}(\mathbb{T}) \\
& \wedge 0 \neq 1 \\
& \wedge (\forall \mathbf{a} \in \mathbb{T}) 0 \cdot \mathbf{a} = \mathbf{a} \cdot 0 = 0 \\
& \wedge (\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{T}) \\
& \quad \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \\
& \quad \wedge (\mathbf{b} + \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a} \\
\text{CommutativeSemiring}(\mathbb{T}) & \triangleq \\
& \text{Semiring}(\mathbb{T}) \\
& \wedge \text{commutative}(\cdot) \\
\text{Ring}(\mathbb{T}) & \triangleq \\
& \text{AdditiveGroup}(\mathbb{T}) \\
& \wedge \text{Semiring}(\mathbb{T}) \\
\text{CommutativeRing}(\mathbb{T}) & \triangleq
\end{aligned}$$

```

    AdditiveGroup(T)
  ∧ CommutativeSemiring(T)
Semimodule(T, S) ≐
  AdditiveMonoid(T)
  ∧ CommutativeSemiring(S)
  ∧ · : S × T → T
  ∧ (∀α, β ∈ S)(∀a, b ∈ T)
    α · (β · a) = (α · β) · a
    (α + β) · a = α · a + β · a
    α · (a + b) = α · a + α · b
    1 · a = a

Module(T, S) ≐
  Semimodule(T, S)
  ∧ AdditiveGroup(T)
  ∧ Ring(S)

OrderedAdditiveSemigroup(T) ≐
  AdditiveSemigroup(T)
  ∧ TotallyOrdered(T)
  ∧ (∀a, b, c ∈ T) a < b ⇒ a + c < b + c

OrderedAdditiveMonoid(T) ≐
  OrderedAdditiveSemigroup(T)
  ∧ AdditiveMonoid(T)

OrderedAdditiveGroup(T) ≐
  OrderedAdditiveMonoid(T)
  ∧ AdditiveGroup(T)

CancellableMonoid(T) ≐
  OrderedAdditiveMonoid(T)
  ∧ - : T × T → T
  ∧ (∀a, b ∈ T) b ≤ a ⇒ a - b is defined ∧ (a - b) + b = a

template<typename T>
  requires(CancellableMonoid(T))
T slow_remainder(T a, T b)
{
  // Precondition: a ≥ 0 ∧ b > 0
  while (b <= a) a = a - b;
  return a;
}

ArchimedeanMonoid(T) ≐
  CancellableMonoid(T)
  ∧ (∀a, b ∈ T) (a ≥ 0 ∧ b > 0) ⇒ slow_remainder(a, b) terminates
  ∧ QuotientType : ArchimedeanMonoid → Integer

```

```

HalvableMonoid(T)  $\triangleq$ 
  ArchimedeanMonoid(T)
   $\wedge$  half : T  $\rightarrow$  T
   $\wedge$  ( $\forall a, b \in T$ ) ( $b > 0 \wedge a = b + b$ )  $\Rightarrow$  half(a) = b

template<typename T>
  requires(ArchimedeanMonoid(T))
T subtractive_gcd_nonzero(T a, T b)
{
  // Precondition: a > 0  $\wedge$  b > 0
  while (true) {
    if (b < a)      a = a - b;
    else if (a < b) b = b - a;
    else           return a;
  }
}

EuclideanMonoid(T)  $\triangleq$ 
  ArchimedeanMonoid(T)
   $\wedge$  ( $\forall a, b \in T$ ) ( $a > 0 \wedge b > 0$ )  $\Rightarrow$  subtractive_gcd_nonzero(a, b) terminates

EuclideanSemiring(T)  $\triangleq$ 
  CommutativeSemiring(T)
   $\wedge$  NormType : EuclideanSemiring  $\rightarrow$  Integer
   $\wedge$  w : T  $\rightarrow$  NormType(T)
   $\wedge$  ( $\forall a \in T$ ) w(a)  $\geq$  0
   $\wedge$  ( $\forall a \in T$ ) w(a) = 0  $\Leftrightarrow$  a = 0
   $\wedge$  ( $\forall a, b \in T$ ) b  $\neq$  0  $\Rightarrow$  w(a  $\cdot$  b)  $\geq$  w(a)
   $\wedge$  remainder : T  $\times$  T  $\rightarrow$  T
   $\wedge$  quotient : T  $\times$  T  $\rightarrow$  T
   $\wedge$  ( $\forall a, b \in T$ ) b  $\neq$  0  $\Rightarrow$  a = quotient(a, b)  $\cdot$  b + remainder(a, b)
   $\wedge$  ( $\forall a, b \in T$ ) b  $\neq$  0  $\Rightarrow$  w(remainder(a, b)) < w(b)

EuclideanSemimodule(T, S)  $\triangleq$ 
  Semimodule(T, S)
   $\wedge$  remainder : T  $\times$  T  $\rightarrow$  T
   $\wedge$  quotient : T  $\times$  T  $\rightarrow$  S
   $\wedge$  ( $\forall a, b \in T$ ) b  $\neq$  0  $\Rightarrow$  a = quotient(a, b)  $\cdot$  b + remainder(a, b)
   $\wedge$  ( $\forall a, b \in T$ ) (a  $\neq$  0  $\vee$  b  $\neq$  0)  $\Rightarrow$  gcd(a, b) terminates

template<typename T, typename S>
  requires(EuclideanSemimodule(T, S))
T gcd(T a, T b)
{
  // Precondition:  $\neg$ (a = 0  $\wedge$  b = 0)
  while (true) {
    if (b == T(0)) return a;
    a = remainder(a, b);
  }
}

```

```

        if (a == T(0)) return b;
        b = remainder(b, a);
    }
}

```

$ArchimedeanGroup(T) \triangleq$
 $ArchimedeanMonoid(T)$
 $\wedge AdditiveGroup(T)$

$DiscreteArchimedeanSemiring(T) \triangleq$
 $CommutativeSemiring(T)$
 $\wedge ArchimedeanMonoid(T)$
 $\wedge (\forall a, b, c \in T) a < b \wedge 0 < c \Rightarrow a \cdot c < b \cdot c$
 $\wedge \neg(\exists a \in T) 0 < a < 1$

$NonnegativeDiscreteArchimedeanSemiring(T) \triangleq$
 $DiscreteArchimedeanSemiring(T)$
 $\wedge (\forall a \in T) 0 \leq a$

$DiscreteArchimedeanRing(T) \triangleq$
 $DiscreteArchimedeanSemiring(T)$
 $\wedge AdditiveGroup(T)$

Chapter 6: Iterators

$Readable(T) \triangleq$
 $Regular(T)$
 $\wedge ValueType : Readable \rightarrow Regular$
 $\wedge source : T \rightarrow ValueType(T)$

$Iterator(T) \triangleq$
 $Regular(T)$
 $\wedge DistanceType : Iterator \rightarrow Integer$
 $\wedge successor : T \rightarrow T$
 $\wedge successor$ is not necessarily regular

property($I : Iterator$)
 $weak_range : I \times DistanceType(I)$
 $(f, n) \mapsto (\forall i \in DistanceType(I))$
 $(0 \leq i \leq n) \Rightarrow successor^i(f)$ is defined

property($I : Iterator$)
 $counted_range : I \times DistanceType(I)$
 $(f, n) \mapsto weak_range(f, n) \wedge$
 $(\forall i, j \in DistanceType(I)) (0 \leq i < j \leq n) \Rightarrow$
 $successor^i(f) \neq successor^j(f)$

property($I : \text{Iterator}$)
bounded_range : $I \times I$
 $(f, l) \mapsto (\exists k \in \text{DistanceType}(I)) \text{counted_range}(f, k) \wedge \text{successor}^k(f) = l$

property($I : \text{Readable}$)
requires(**Iterator**(I))
readable_bounded_range : $I \times I$
 $(f, l) \mapsto \text{bounded_range}(f, l) \wedge (\forall i \in [f, l]) \text{source}(i) \text{ is defined}$

property($Op : \text{BinaryOperation}$)
partially_associative : Op
 $op \mapsto (\forall a, b, c \in \text{Domain}(op))$
 If $op(a, b)$ and $op(b, c)$ are defined,
 $op(op(a, b), c)$ and $op(a, op(b, c))$ are defined
 and are equal.

ForwardIterator(T) \triangleq
 $\text{Iterator}(T)$
 $\wedge \text{regular_unary_function}(\text{successor})$

IndexedIterator(T) \triangleq
 $\text{ForwardIterator}(T)$
 $\wedge + : T \times \text{DistanceType}(T) \rightarrow T$
 $\wedge - : T \times T \rightarrow \text{DistanceType}(T)$
 $\wedge +$ takes constant time
 $\wedge -$ takes constant time

BidirectionalIterator(T) \triangleq
 $\text{ForwardIterator}(T)$
 $\wedge \text{predecessor} : T \rightarrow T$
 $\wedge \text{predecessor}$ takes constant time
 $\wedge (\forall i \in T) \text{successor}(i) \text{ is defined} \Rightarrow$
 $\text{predecessor}(\text{successor}(i)) \text{ is defined and equals } i$
 $\wedge (\forall i \in T) \text{predecessor}(i) \text{ is defined} \Rightarrow$
 $\text{successor}(\text{predecessor}(i)) \text{ is defined and equals } i$

RandomAccessIterator(T) \triangleq
 $\text{IndexedIterator}(T) \wedge \text{BidirectionalIterator}(T)$
 $\wedge \text{TotallyOrdered}(T)$
 $\wedge (\forall i, j \in T) i < j \Leftrightarrow i \prec j$
 $\wedge \text{DifferenceType} : \text{RandomAccessIterator} \rightarrow \text{Integer}$
 $\wedge + : T \times \text{DifferenceType}(T) \rightarrow T$
 $\wedge - : T \times \text{DifferenceType}(T) \rightarrow T$
 $\wedge - : T \times T \rightarrow \text{DifferenceType}(T)$
 $\wedge <$ takes constant time
 $\wedge -$ between an iterator and an integer takes constant time

Chapter 7: Coordinate Structures

```

BifurcateCoordinate(T)  $\triangleq$ 
  Regular(T)
   $\wedge$  WeightType : BifurcateCoordinate  $\rightarrow$  Integer
   $\wedge$  empty : T  $\rightarrow$  bool
   $\wedge$  has_left_successor : T  $\rightarrow$  bool
   $\wedge$  has_right_successor : T  $\rightarrow$  bool
   $\wedge$  left_successor : T  $\rightarrow$  T
   $\wedge$  right_successor : T  $\rightarrow$  T
   $\wedge$  ( $\forall i, j \in T$ ) (left_successor(i) = j  $\vee$  right_successor(i) = j)  $\Rightarrow$   $\neg$ empty(j)

```

property(C : BifurcateCoordinate)

tree : C

$x \mapsto$ the descendants of x form a tree

```

BidirectionalBifurcateCoordinate(T)  $\triangleq$ 
  BifurcateCoordinate(T)
   $\wedge$  has_predecessor : T  $\rightarrow$  bool
   $\wedge$  ( $\forall i \in T$ )  $\neg$ empty(i)  $\Rightarrow$  has_predecessor(i) is defined
   $\wedge$  predecessor : T  $\rightarrow$  T
   $\wedge$  ( $\forall i \in T$ ) has_left_successor(i)  $\Rightarrow$ 
    predecessor(left_successor(i)) is defined and equals i
   $\wedge$  ( $\forall i \in T$ ) has_right_successor(i)  $\Rightarrow$ 
    predecessor(right_successor(i)) is defined and equals i
   $\wedge$  ( $\forall i \in T$ ) has_predecessor(i)  $\Rightarrow$ 
    is_left_successor(i)  $\vee$  is_right_successor(i)

```

```

template<typename T>

```

```

  requires(BidirectionalBifurcateCoordinate(T))

```

```

bool is_left_successor(T j)

```

```

{

```

```

  // Precondition: has_predecessor(j)

```

```

  T i = predecessor(j);

```

```

  return has_left_successor(i) && left_successor(i) == j;

```

```

}

```

```

template<typename T>

```

```

  requires(BidirectionalBifurcateCoordinate(T))

```

```

bool is_right_successor(T j)

```

```

{

```

```

  // Precondition: has_predecessor(j)

```

```

  T i = predecessor(j);

```

```

  return has_right_successor(i) && right_successor(i) == j;

```

```

}

```

property(C : Readable)

requires(BifurcateCoordinate(C))

readable_tree : C
 $x \mapsto \text{tree}(x) \wedge (\forall y \in C) \text{reachable}(x, y) \Rightarrow \text{source}(y) \text{ is defined}$

Chapter 8: Coordinates with Mutable Successors

ForwardLinker(S) \triangleq

IteratorType : *ForwardLinker* \rightarrow *ForwardIterator*
 \wedge Let I = IteratorType(S) in:
 $(\forall s \in S) (s : I \times I \rightarrow \text{void})$
 $\wedge (\forall s \in S) (\forall i, j \in I) \text{if } \text{successor}(i) \text{ is defined,}$
then $s(i, j)$ establishes $\text{successor}(i) = j$

BackwardLinker(S) \triangleq

IteratorType : *BackwardLinker* \rightarrow *BidirectionalIterator*
 \wedge Let I = IteratorType(S) in:
 $(\forall s \in S) (s : I \times I \rightarrow \text{void})$
 $\wedge (\forall s \in S) (\forall i, j \in I) \text{if } \text{predecessor}(j) \text{ is defined,}$
then $s(i, j)$ establishes $i = \text{predecessor}(j)$

BidirectionalLinker(S) \triangleq *ForwardLinker*(S) \wedge *BackwardLinker*(S)

property(I : *Iterator*)

disjoint : $I \times I \times I \times I$
 $(f0, l0, f1, l1) \mapsto (\forall i \in I) \neg(i \in [f0, l0] \wedge i \in [f1, l1])$

LinkedBifurcateCoordinate(T) \triangleq

BifurcateCoordinate(T)
 \wedge set_left_successor : $T \times T \rightarrow \text{void}$
 $(i, j) \mapsto \text{establishes } \text{left_successor}(i) = j$
 \wedge set_right_successor : $T \times T \rightarrow \text{void}$
 $(i, j) \mapsto \text{establishes } \text{right_successor}(i) = j$

EmptyLinkedBifurcateCoordinate(T) \triangleq

LinkedBifurcateCoordinate(T)
 \wedge empty(T())¹
 \wedge $\neg \text{empty}(i) \Rightarrow$
left_successor(i) and right_successor(i) are defined
 \wedge $\neg \text{empty}(i) \Rightarrow$
 $(\neg \text{has_left_successor}(i) \Leftrightarrow \text{empty}(\text{left_successor}(i)))$
 \wedge $\neg \text{empty}(i) \Rightarrow$
 $(\neg \text{has_right_successor}(i) \Leftrightarrow \text{empty}(\text{right_successor}(i)))$

¹In other words, empty is true on the default constructed value and possibly on other values as well.

Chapter 9: Copying

$Writable(T) \triangleq$
 $ValueType : Writable \rightarrow Regular$
 $\wedge (\forall x \in T) (\forall v \in ValueType(T)) \text{sink}(x) \leftarrow v$ is a well-formed statement

property($T : Writable, U : Readable$)
requires($ValueType(T) = ValueType(U)$)
 $aliased : T \times U$
 $(x, y) \mapsto \text{sink}(x)$ is defined \wedge
 $\text{source}(y)$ is defined \wedge
 $(\forall v \in ValueType(T)) \text{sink}(x) \leftarrow v$ establishes $\text{source}(y) = v$

$Mutable(T) \triangleq$
 $Readable(T) \wedge Writable(T)$
 $\wedge (\forall x \in T) \text{sink}(x)$ is defined $\Leftrightarrow \text{source}(x)$ is defined
 $\wedge (\forall x \in T) \text{sink}(x)$ is defined $\Rightarrow aliased(x, x)$
 $\wedge \text{deref} : T \rightarrow ValueType(T) \&$
 $\wedge (\forall x \in T) \text{sink}(x)$ is defined $\Leftrightarrow \text{deref}(x)$ is defined

property($I : Writable$)
requires($Iterator(I)$)
 $writable_bounded_range : I \times I$
 $(f, l) \mapsto bounded_range(f, l) \wedge (\forall i \in [f, l]) \text{sink}(i)$ is defined

$writable_weak_range$ and $writable_counted_range$ are defined similarly.

property($I : Mutable$)
requires($ForwardIterator(I)$)
 $mutable_bounded_range : I \times I$
 $(f, l) \mapsto bounded_range(f, l) \wedge (\forall i \in [f, l]) \text{sink}(i)$ is defined

$mutable_weak_range$ and $mutable_counted_range$ are defined similarly.

property($I : Readable, O : Writable$)
requires($Iterator(I) \wedge Iterator(O)$)
 $not_overlapped_forward : I \times I \times O \times O$
 $(f_i, l_i, f_o, l_o) \mapsto$
 $readable_bounded_range(f_i, l_i) \wedge$
 $writable_bounded_range(f_o, l_o) \wedge$
 $(\forall k_i \in [f_i, l_i])(\forall k_o \in [f_o, l_o])$
 $aliased(k_o, k_i) \Rightarrow k_i - f_i \leq k_o - f_o$

property($I : Readable, O : Writable$)
requires($Iterator(I) \wedge Iterator(O)$)
 $not_overlapped_backward : I \times I \times O \times O$
 $(f_i, l_i, f_o, l_o) \mapsto$
 $readable_bounded_range(f_i, l_i) \wedge$
 $writable_bounded_range(f_o, l_o) \wedge$

$$(\forall k_i \in [f_i, l_i])(\forall k_o \in [f_o, l_o]) \\ \text{aliased}(k_o, k_i) \Rightarrow l_i - k_i \leq l_o - k_o$$

property($I : \text{Readable}, O : \text{Writable}$)
requires($\text{Iterator}(I) \wedge \text{Iterator}(O)$)

$\text{not_overlapped} : I \times I \times O \times O$

$$(f_i, l_i, f_o, l_o) \mapsto \\ \text{readable_bounded_range}(f_i, l_i) \wedge \\ \text{writable_bounded_range}(f_o, l_o) \wedge \\ (\forall k_i \in [f_i, l_i])(\forall k_o \in [f_o, l_o]) \neg \text{aliased}(k_o, k_i)$$

property($T : \text{Writable}, U : \text{Writable}$)
requires($\text{ValueType}(T) = \text{ValueType}(U)$)

$\text{write_aliased} : T \times U$

$$(x, y) \mapsto \text{sink}(x) \text{ is defined} \wedge \text{sink}(y) \text{ is defined} \wedge \\ (\forall V \in \text{Readable})(\forall v \in V) \text{ aliased}(x, v) \Leftrightarrow \text{aliased}(y, v)$$

property($O_0 : \text{Writable}, O_1 : \text{Writable}$)
requires($\text{Iterator}(O_0) \wedge \text{Iterator}(O_1)$)

$\text{not_write_overlapped} : O_0 \times O_0 \times O_1 \times O_1$

$$(f_0, l_0, f_1, l_1) \mapsto \\ \text{writable_bounded_range}(f_0, l_0) \wedge \\ \text{writable_bounded_range}(f_1, l_1) \wedge \\ (\forall k_0 \in [f_0, l_0])(\forall k_1 \in [f_1, l_1]) \neg \text{write_aliased}(k_0, k_1)$$

property($I : \text{Readable}, O : \text{Writable}, N : \text{Integer}$)
requires($\text{Iterator}(I) \wedge \text{Iterator}(O)$)

$\text{backward_offset} : I \times I \times O \times O \times N$

$$(f_i, l_i, f_o, l_o, n) \mapsto \\ \text{readable_bounded_range}(f_i, l_i) \wedge \\ n \geq 0 \wedge \\ \text{writable_bounded_range}(f_o, l_o) \wedge \\ (\forall k_i \in [f_i, l_i])(\forall k_o \in [f_o, l_o]) \\ \text{aliased}(k_o, k_i) \Rightarrow k_i - f_i + n \leq k_o - f_o$$

property($I : \text{Readable}, O : \text{Writable}, N : \text{Integer}$)
requires($\text{Iterator}(I) \wedge \text{Iterator}(O)$)

$\text{forward_offset} : I \times I \times O \times O \times N$

$$(f_i, l_i, f_o, l_o, n) \mapsto \\ \text{readable_bounded_range}(f_i, l_i) \wedge \\ n \geq 0 \wedge \\ \text{writable_bounded_range}(f_o, l_o) \wedge \\ (\forall k_i \in [f_i, l_i])(\forall k_o \in [f_o, l_o]) \\ \text{aliased}(k_o, k_i) \Rightarrow l_i - k_i + n \leq l_o - k_o$$

Index

- (product)
 - in multiplicative semigroup, 5
 - in semimodule, 6
- AdditiveGroup* concept, 5
- AdditiveMonoid* concept, 5
- AdditiveSemigroup* concept, 4
- algorithm
 - gcd, 8
 - is_left_successor, 10
 - is_right_successor, 10
 - slow_remainder, 6
 - subtractive_gcd_nonzero, 7
- aliased property, 12
- ArchimedeanGroup* concept, 8
- ArchimedeanMonoid* concept, 6
- associative operation, 9
- associative property, 2
 - partially_associative, 9
- asymmetric property, 3
- backward_offset property, 13
- BackwardLinker* concept, 11
- begin
 - for *Linearizable*, 14
- BidirectionalBifurcateCoordinate* concept, 10
- BidirectionalIterator* concept, 9
- BidirectionalLinker* concept, 11
- BifurcateCoordinate* concept, 10
- binary_scale_down_nonnegative, 3
- binary_scale_up_nonnegative, 3
- BinaryOperation* concept, 2
- bounded_range property, 9
- CancellableMonoid* concept, 6
- commutative property, 4
- CommutativeRing* concept, 5
- CommutativeSemiring* concept, 5
- concept
 - AdditiveGroup*, 5
 - AdditiveMonoid*, 5
 - AdditiveSemigroup*, 4
 - ArchimedeanGroup*, 8
 - ArchimedeanMonoid*, 6
 - BackwardLinker*, 11
 - BidirectionalBifurcateCoordinate*, 10
 - BidirectionalIterator*, 9
 - BidirectionalLinker*, 11
 - BifurcateCoordinate*, 10
 - BinaryOperation*, 2
 - CancellableMonoid*, 6
 - CommutativeRing*, 5
 - CommutativeSemiring*, 5
 - DiscreteArchimedeanRing*, 8
 - DiscreteArchimedeanSemiring*, 8
 - EmptyLinkedBifurcateCoordinate*, 11
 - EuclideanMonoid*, 7
 - EuclideanSemimodule*, 7
 - EuclideanSemiring*, 7
 - ForwardIterator*, 9
 - ForwardLinker*, 11
 - FunctionalProcedure*, 1
 - HalvableMonoid*, 7
 - HomogeneousFunction*, 1
 - HomogeneousPredicate*, 2
 - IndexedIterator*, 9
 - Integer*, 2
 - Iterator*, 8
 - Linearizable*, 14
 - LinkedBifurcateCoordinate*, 11
 - Module*, 6
 - MultiplicativeGroup*, 5
 - MultiplicativeMonoid*, 5
 - MultiplicativeSemigroup*, 5
 - Mutable*, 12
 - NonnegativeDiscreteArchimedeanSemiring*, 8
 - Operation*, 2
 - OrderedAdditiveGroup*, 6
 - OrderedAdditiveMonoid*, 6
 - OrderedAdditiveSemigroup*, 6
 - Predicate*, 2
 - RandomAccessIterator*, 9
 - Readable*, 8
 - Regular*, 1

- Relation*, 3
- Ring*, 5
- Semimodule*, 6
- Semiring*, 5
- Sequence*, 14
- TotallyOrdered*, 4
- Transformation*, 2
- UnaryFunction*, 1
- UnaryPredicate*, 2
- Writable*, 12
- counted_range property, 8
- deref, 12
- DifferenceType type function, 9
- DiscreteArchimedeanRing* concept, 8
- DiscreteArchimedeanSemiring* concept, 8
- disjoint property, 11
- DistanceType type function, 2, 8
- Domain type function, 1
- empty
 - for *Linearizable*, 14
- EmptyLinkedBifurcateCoordinate* concept, 11
- end
 - for *Linearizable*, 14
- equivalence property, 3
- EuclideanMonoid* concept, 7
- EuclideanSemimodule* concept, 7
- EuclideanSemiring* concept, 7
- even, 3
- forward_offset property, 13
- ForwardIterator* concept, 9
- ForwardLinker* concept, 11
- FunctionalProcedure* concept, 1
- gcd algorithm, 8
- half_nonnegative, 2
- HalvableMonoid* concept, 7
- HomogeneousFunction* concept, 1
- HomogeneousPredicate* concept, 2
- identity_element property, 4
- IndexedIterator* concept, 9
- Integer* concept, 2
- inverse_operation property, 4
- is_left_successor algorithm, 10
- is_right_successor algorithm, 10
- Iterator* concept, 8
- IteratorType type function, 11, 14
- Linearizable* concept, 14
- LinkedBifurcateCoordinate* concept, 11
- mergeable property, 14
- Module* concept, 6
- MultiplicativeGroup* concept, 5
- MultiplicativeMonoid* concept, 5
- MultiplicativeSemigroup* concept, 5
- Mutable* concept, 12
- mutable_bounded_range property, 12
- mutable_counted_range property, 12
- mutable_weak_range property, 12
- negative, 3
- NonnegativeDiscreteArchimedeanSemiring* concept, 8
- not_overlapped property, 13
- not_overlapped_backward property, 12
- not_overlapped_forward property, 12
- not_write_overlapped property, 13
- odd, 3
- one, 3
- Operation* concept, 2
- OrderedAdditiveGroup* concept, 6
- OrderedAdditiveMonoid* concept, 6
- OrderedAdditiveSemigroup* concept, 6
- partially_associative property, 9
- positive, 3
- predecessor
 - of integer, 2
 - of iterator, 9
- Predicate* concept, 2
- product (·)
 - in multiplicative semigroup, 5
 - in semimodule, 6
- property
 - aliased, 12
 - associative, 2

- asymmetric, 3
- backward_offset, 13
- bounded_range, 9
- commutative, 4
- counted_range, 8
- disjoint, 11
- equivalence, 3
- forward_offset, 13
- identity_element, 4
- inverse_operation, 4
- mergeable, 14
- mutable_bounded_range, 12
- mutable_counted_range, 12
- mutable_weak_range, 12
- not_overlapped, 13
- not_overlapped_backward, 12
- not_overlapped_forward, 12
- not_write_overlapped, 13
- partially_associative, 9
- readable_bounded_range, 9
- readable_tree, 11
- reflexive, 3
- regular_unary_function, 1
- strict, 3
- symmetric, 3
- total_ordering, 4
- transitive, 3
- tree, 10
- weak_ordering, 4
- weak_range, 8
- writable_bounded_range, 12
- writable_counted_range, 12
- writable_weak_range, 12
- write_alias, 13

quotient

- in Euclidean semimodule, 7
- in Euclidean semiring, 7

QuotientType type function, 6

RandomAccessIterator concept, 9

Readable concept, 8

readable_bounded_range property, 9

readable_tree property, 11

reflexive property, 3

Regular concept, 1

regular_unary_function property, 1

Relation concept, 3

remainder

- in Euclidean semimodule, 7
- in Euclidean semiring, 7

Ring concept, 5

Semimodule concept, 6

Semiring concept, 5

Sequence concept, 14

sink, 12

size

- for *Linearizable*, 14

SizeType type function, 14

slow_remainder algorithm, 6

source, 8

strict property, 3

subtractive_gcd_nonzero algorithm, 7

successor

- of integer, 2
- of iterator, 8

symmetric property, 3

total_ordering property, 4

TotallyOrdered concept, 4

Transformation concept, 2

transitive property, 3

tree property, 10

twice, 2

type function

- DifferenceType, 9
- DistanceType, 2, 8
- Domain, 1
- IteratorType, 11, 14
- QuotientType, 6
- SizeType, 14
- ValueType, 8, 12, 14
- WeightType, 10

UnaryFunction concept, 1

UnaryPredicate concept, 2

ValueType type function, 8, 12, 14

weak_ordering property, 4

weak_range property, 8

WeightType type function, 10

Writable concept, 12
writable_bounded_range property, 12
writable_counted_range property, 12
writable_weak_range property, 12
write_aliased property, 13

zero, 3